JOURNAL OF COMBINATORIAL THEORY 9, 108–116 (1969)

# Some Combinatorial Properties of Transformations and Their Connections with the Theory of Graphs

J. DÉNES

*Hungarian Academy of Sciences, Central Research Institute for Physics Budapest 114, P.O.B. 49, Hungary*

*Communicated by Claude Berge*

ABSTRACT

Although many results concerning permutations and permutation groups are known, less attention has been paid to transformations and transformation semigroups. It is true that every abstract group is isomorphic to a permutation group, so that with respect to structure there is not difference between abstract groups and permutation groups. Similarly every abstract semigroup is isomorphic to a transformation semigroup; this has led the author to write some papers on the subject [4, 5]. We restrict ourselves to the finite case, and the aim of this paper is to obtain results in this field by a one-to-one correspondence between transformations and directed graphs. The main results of this paper are as follows: (1) Generalization of the Cauchy formula concerning the number of permutations of degree $n$ with prescribed lengths of cycles. (2) Determination of the number of element triples which are generating systems of the symmetric semigroup of degree $n$, i.e., the semigroup containing every transformation of degree $n$. (3) Determination of the expected value of the degree of the main permutation of a random transformation of degree $n$.

Before entering on the subject some definitions which are needed throughout this paper will be given, emphasizing the similarity between permutations and transformations.

A permutation of degree $n$ is a mapping of a set of $n$ elements onto itself.

A transformation of degree $n$ can be defined in the same way, only the word "onto" must be replaced by "into."

Permutations and transformations may be written in matrix form, as follows:

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & & a_n \end{pmatrix}$$

where the $a_i$ ($i = 1, 2,..., n$) are all distinct, if $\alpha$ is a permutation; otherwise $\alpha$ is a transformation.

The number of the distinct elements of the set $\{a_1, a_2, ..., a_n\}$ is called the defect of $\alpha$.

A transformation is called singular if exactly one element of the set is not mapped onto itself.

Clearly a transformation is a permutation if and only if the number of defects in it is $n$. Obviously the number of defects in a singular transformation is $n - 1$.

A singular transformation of degree $n$ is one in which the elements $1, 2, ..., i-1, i+1, ..., n$ are mapped onto themselves and $i$ is mapped onto $j$; this singular transformation will be denoted by $\binom{i}{j}$.

To every transformation of degree $n$ there corresponds an $n \times n$ matrix as follows: if $i$ is mapped onto $j$ by the given transformation, then the entry in $i$-th row and $j$-th column is a 1; all other entries being 0, consequently every row contains exactly one symbol 1.

Another type of representation of transformations is the graph representation. To every transformation of degree $n$ there corresponds uniquely a directed graph with $n$ labeled vertices in such a way that the vertices are labeled by the natural numbers $1, 2, ..., n$ such that, if the transformation maps $i$ to $j$, then the corresponding edge is directed from $i$ to $j$ (see [4, 5, 10]).

It is easy to see that a directed graph corresponds to a transformation if and only if each of its connected disjoint subgraphs contains a single circuit and directed trees (see [10]). Furthermore the edges of the circuit are directed cyclically, and the edges of the trees are directed toward the corresponding cycle. In [10] one can find the result in generalized form extending to the infinite case.

Such graphs with $n$ vertices will be called $F(n)$ graphs. An immediate consequence of this correspondence is that the number of labeled $F(n)$ graphs is $n^n$. $F(n)$ graphs with $n$ edges have been ennumerated by Katz [9] and Rényi [13].

Two directed graphs are called isomorphic if there is a one-to-one correspondence between their sets of vertices which preserves their directed edges. The number of non-isomorphic $F(n)$ graphs without slings was obtained by Davis [2] and Harary [7].

A transformation corresponding to a single connected component of an $F(n)$ graph is conveniently called a generalized cycle, since in the case of a permutation it is the conventional cycle. Every transformation may be uniquely represented as the product of disjoint generalized cycles in the same way as a permutation may be uniquely represented as the product of disjoint cycles. "Uniquely" is understood here to exclude the arrangement of the cycles.

On deleting the trees from an $F(n)$ graph one obtains a special $F(k)$ graph

($k \leqslant n$) containing only cycles. The transformation corresponding to this graph is a permutation; let us call it the main permutation of the original transformation [4, 5].

Erdös and Turán [6] introduced the concept of statistical group theory, by which they mean the study of those properties of certain complexes of a "larger" group which are shared by "most" of these complexes. The group considered in [6] is $S_n$, i.e., the group containing every permutation of degree $n$.

A similar investigation concerning $F_n$, i.e., a semigroup containing every transformation of degree $n$, will be described in this paper (see also [5, 16]).

LEMMA 1. *Arbitrary transformation can be represented as*

(a) *the product of disjoint generalized cycles and singular transformations,*
(b) *the product of transpositions and singular transformations.*

LEMMA 2. *$F_n$ is generated by one of its arbitrary singular transformation $\binom{i}{j}$ and $S_n$, i.e.,*

$$F_n = \left\{\binom{i}{j}, S_n\right\}.$$

LEMMA 3. *The minimum number of elements contained in a generating system of $F_n$ ($n \geqslant 3$) is three.*

LEMMA 4. *If $S_n = \{\alpha, \beta\}$ then $F_n = \{\alpha, \beta, \gamma\}$ holds if and only if the number of defects of $\gamma$ is $n - 1$.*

The proofs of the four lemmas are not given here, since they are published in [4].

Let us denote by $M_n$ the number of distinct generating systems of $S_n$ containing two elements; similarly let $K_n$ be the number of generating systems of $F_n$ containing the minimum number of elements.

THEOREM 1. $K_n = n!\binom{n}{2} M_n$.

By Lemma 3 $F_n$ is generated by three elements at least. Lemma 4 states that two of the generating elements are contained in $S_n$ and one of them is of $n - 1$ defects. It remains to prove that the number of transformations with $n - 1$ defects contained in $F_n$ is $n!\binom{n}{2}$.

To prove this statement it is convenient to consider the graph representation of the transformation. It is obvious that, if a transformation has $n - 1$ defects, then the corresponding graph is an $F_n$ graph with a single end-point. Let us consider those transformations with $n - 1$ defects whose degree of their main permutations is $n - i$. The number of vertices in the chain is $i$. There exist $\binom{n}{n-i}$ ways of choosing the vertices contained in the main permutation, such that in each case $(n - i)!$ distinct main permutation may be constructed; there are $i!$ ways of arranging the vertices of the chain, and the chain may be joined by $n - i$ distinct vertices of the main permutation. Hence the number of labeled $F(n)$ graphs with a single end-point is

$$\sum_{i=1}^{n-1} \binom{n}{n-i} (n-i)!\, i!(n-i) = \sum_{i=1}^{n-1} n!(n-i) = n! \binom{n}{2}.$$

Let us call two generating systems $T_1$, $T_2$ of $F_n$ independent if there exists in $S_n$ no element $\rho$ for which

$$\rho T_1 \rho^{-1} = \{\rho\alpha_1\rho^{-1}, \rho\alpha_2\rho^{-1}, \dots, \rho\alpha_s\rho^{-1}\} = T_2,$$

where $\alpha_1, \alpha_2, \dots, \alpha_s$ denote the elements of $T_1$ and the braces denote the set of the included elements.

Piccard [11] has obtained several results for $M_n$. For $n \leqslant 7$, $M_n$ were determined by her. We have used these results, and our theorem, in obtaining Table I.

TABLE I

| $n$ | $M_n$ | $K_n$ |
|---|---|---|
| 2 | 1 | 2 |
| 3 | 9 | 162 |
| 4 | 108 | 15552 |
| 5 | 3420 | $4104 \times 10^3$ |
| 6 | 114480 | $1236384 \times 10^3$ |
| 7 | 7786800 | $824154912 \times 10^3$ |

TABLE II

| $n$ | $\dfrac{M_n}{\binom{n!}{2}} = P_n$ | $\dfrac{K_n}{\binom{n^n}{3}} = P_n^*$ |
|---|---|---|
| 3 | 0,6 | 0,055 |
| 4 | 0,39 | 0,0056 |
| 5 | 0,46 | 0,00081 |
| 6 | 0,44 | 0,000073 |
| 7 | 0,61 | 0,0000089 |

$P_n$ denotes the probability that, choosing two elements $\alpha, \beta$ at random from $S_n$, the whole of $S_n$ is generated by $\alpha, \beta$.

$P^*$ denotes the probability that, choosing three elements $\alpha, \beta, \gamma$ at random from $F_n$, the whole of $F_n$ is generated by $\alpha, \beta, \gamma$.

One can observe that $P_n$ is surprisingly large. Recently Dixon proved, that $P_n \to 0{,}75$, if $n \to \infty$, See [15]. Ulam, in his book [14], raised the problem of finding the mean values of $0\{\alpha, \beta\}$ $(\alpha, \beta \in S_n)$ and $0\{\alpha', \beta'\}(\alpha', \beta' \in F_n)$. He asked for the result of a comparison between the mean values of $0\{\alpha, \beta\}$ and of $0\{\alpha', \beta'\}$.

The author's conjecture is that the mean value of $0\{\alpha', \beta'\}$ is less than the expected value of $0\{\alpha, \beta\}$. The author's conjecture is based on a result which will be published in [16].

Another approach to Ulam's problem will be given.

As in the case of groups, also in the case of semigroups, one can define the order of an element, i.e., the number of elements of the substructure generated by this element. Trivially, the definition which follows is equal to the former definition. Let $\alpha$ be an arbitrary element of the semigroup $S$ (by Cayley's theorem we may suppose without loss of generality that $\alpha$ is a transformation); the order of $\alpha$ is the lowest exponent $r$ $(r > 0)$ for which there exists a lower exponent $s$ $(r \geqslant s \geqslant 0)$ such that the equality $\alpha^{r+1} = \alpha^s$ holds.

THEOREM 2. *Let $k$ denote the order of the main permutation of a transformation $\alpha$, i.e., the least common multiple of its cycle lengths, and let $h$ be the maximum height of the tree structures (i.e., the maximum length of the path connecting the vertex contained in the circuit with the end-point). Then the order of $\alpha$ is $r = k + h - 1$ if $\alpha$ is not a permutation.*

PROOF: Let us consider the graph corresponding to $\alpha$ and let us label the components and the vertices within the components. The length of the circuit in the $i$-th component will be denoted by $l_i$; and the distance between $j$-th point of the $i$-th component and the circuit will be denoted by $m_{ij}$.

Then $\max\{m_{ij}\} = h$ and the least common multiple of $l_i$ $(i = 1, 2, \ldots)$ is $k$. Since $\alpha^{r+1} = \alpha^s$, the images in both transformations are the same. As $r + 1 \neq s$ holds, it is necessary that, when applying both transformations, the image of the given element shall be in the main permutation. Let us suppose that the vertex corresponding to the given element is mapped to a vertex being at length $x$ from the common point of its containing tree and the circuit, then

$$x \equiv S - m_{ij} (\mathrm{mod}\ l_i);$$

then

$$x \equiv r + 1 - m_{ij}(\text{mod } l_i);$$

hence

$$S - m_{ij} \equiv r + 1 - m_{ij}(\text{mod } l_i)$$

for arbitrary possible pair of $i$, $j$; thus

$$S \equiv r + 1(\text{mod } l_i)$$

holds for every $i$. One has to find the least $r$; as for $S$ the only restriction is that applying $\alpha^s$, every vertex is mapped to a vertex contained in a certain circuit, and such minimum value of $S$ is $\max_{i,j}(m_{ij}) = h$,

$$h \equiv r + 1(\text{mod } l_i)$$

for every $i$; thus

$$h \equiv r + 1(\text{mod } k).$$

Clearly the least value is

$$r + 1 = h + k$$

since $r + 1 = h$ would imply $r < h = s$, contrary to the original assumption. It follows that the equation

$$r = h + k - 1$$

holds.

REMARK. Another proof of Theorem 2 has been given by Ádám [1].

The first step toward obtaining the mean value of the order of a transformation of degree $n$ is to determine the mean value of the degree of its main permutation.

THEOREM 3. *The mean value of the degree of the main permutation of a random transformation of degree n is $K_m$, where*

$$K_m = \frac{n!}{n^n} \sum_{K=1}^{n} \frac{K}{(n-K)!} \cdot \sum_{l=0}^{K} \binom{K}{l} \frac{(n-K)^{n-2k+l}}{(K-l)!}$$

$$\cdot \sum_{j=0}^{K-l} \left(-\frac{1}{2}\right)^j \binom{k-l}{j} \binom{n-k-1}{K-l-1+j} (K-l+j)!\,(n-k)^{-j}.$$

PROOF: Let us denote by $N_k(n)$ the number of transformations of degree $n$ whose main permutations are of degree $K$ and let $G_t(n)$ denote the number of graphs with $n$ labeled vertices consisting of $t$ connected components each of which is a tree. It is known that

$$G_t(n) = n! \sum_{\substack{\sum_{j=1}^{n} a_j = t \\ \sum_{j=1}^{n} j a_j = n}} \prod_{j=1}^{n} \left(\frac{j^{j-2}}{j!}\right)^{a_j} \cdot \frac{1}{a_j!} \tag{1a}$$

and

$$G_t(n) = \frac{1}{t!} \sum_{j=0}^{t} \left(-\frac{1}{2}\right)^j \binom{t}{j} \binom{n-1}{t+j-1} n^{n-t-j} (t+j)! \tag{1b}$$

hold (see [3] and [12]).

There are $\binom{n}{K}$ ways of choosing $K$ vertices to be contained in the main permutation from the $n$ vertices of a graph, and the number of distinct permutations of these $K$ vertices is $K!$. If $l$ denotes the number of vertices not contained in a cycle then $l = 0, 1, \ldots, K$ and there are $\binom{K}{l}$ possible ways of choosing them. Then, using $n - k$ vertices $K - l$ trees have to be constructed, and so for $N_K(n)$ the following equality holds:

$$N_K(n) = \binom{n}{K} K! \sum_{l=0}^{K} \binom{K}{l} G_{K-l}(n-k). \tag{2}$$

This is easily verified by using the corresponding graphs. The expected value of the degree of the main permutation $K_m$ is then

$$K_m = \sum_{K=1}^{n} \frac{N_K(n)}{n^n} K. \tag{3}$$

As a consequence of (1b) and (2),

$$\begin{aligned} N_K(n) = \binom{n}{K} K! \sum_{l=0}^{K} \binom{K}{l} \frac{(n-k)^{n-2k+l}}{(K-l)!} \\ \cdot \sum_{j=0}^{K-l} \left(-\frac{1}{2}\right)^j \binom{K-l}{j} \binom{n-k-1}{K-l-1+j} (K-l+j)!\,(n-k)^{-j}; \end{aligned} \tag{4}$$

using (3) and (4) the proof follows immediately.

It is well known that, in $S_n$, the number of permutations with the number of cycles $a_i$ of length $i$ is equal to

$$\frac{n!}{1^{a_1} a_1!\, 2^{a_2} a_2! \cdots n^{a_n} a_n!}.$$

This result due to Cauchy.

The next theorem is a generalization of Cauchy's formula to the transformations of degree $n$, i.e., the elements of $F_n$.

LEMMA 5. *Let us denote the number of $F(i)$ graphs of degree $K$ with a single connected component by $N[T(i)]$. Then*

$$N[T(i)] = i^{i-1} + \sum_{j=1}^{[i/2]} \binom{i}{j} j^{j-1}(i-j)^{i-j-1} + i^{i-1} \sum_{j=3}^{i} \binom{i}{j} \frac{j!}{i^j}$$

PROOF: Rényi proved in [13] that the number of connected graphs with $i$ vertices and edges is

$$\frac{1}{2} i^{i-1} \sum_{m=3}^{i} \prod_{j=1}^{m-1} \left(1 - \frac{1}{i}\right)$$

(see also [9]). Using Rényi's formula and taking into consideration that Rényi enumerated graphs are without slings as well as that the number of rooted labeled trees with $i$ vertices is $i^{(i-1)}$, $N[T(i)]$ can be determined as follows:

$$N[T(i)] = i^{i-1} + \sum_{j=1}^{[i/2]} \binom{i}{j} j^{j-1}(i-j)^{i-j-1} + i^{i-1} \sum_{j=3}^{i} \binom{i}{j} \frac{j!}{i^j}$$

THEOREM 4. *If $T_{(a_1,a_2,\ldots,a_n)}$ denotes the number of transformations of degree $n$ with $a_i$ generalized cycles of length $i$, i.e., the number of transformation of type $a_1, a_2, \ldots, a_n$, then*

$$T_{(a_1,a_2,\ldots,a_n)} = n! \prod_{i=1}^{n} \frac{1}{(i!)^{a_i} a_i!} \left[ \sum_{j=1}^{[i/2]} \binom{i}{j} j^{j-1}(i-j)^{i-j-1} + i^{i-1} \sum_{j=3}^{i} \binom{i}{j} \frac{j!}{i^j} \right]^{a_i}$$

PROOF: Denote by $C_n$ the number of ways in which $n$ elements can be distributed to generalized cycles of a transformation of type $a_1, a_2, \ldots, a_n$ regardless of their order within the generalized cycle. Then using the Cauchy formula and elementary arguments one obtains

$$C_n = \frac{n!}{\prod_{i=1}^{n} i^{a_i} a_i!} \cdot \frac{1}{\prod_{i=1}^{n} (i-1)!^{a_i}},$$

$$C_n = \frac{n!}{\prod_{i=1}^{n} i!^{a_i} a_i!}.$$

Using the graph representation of transformations it is obvious that the number of generalized cycles of length $k$ is equal to the number of $T(k)$ graphs with a single connected component, i.e., to $N[T(k)]$. Since

$$T_{(a_1, a_2, \cdots, a_n)} = C_n \prod_{i=1}^{n} N[T(i)]^{a_i}$$

the proof of the theorem follows immediately.

## References


1. A. Ádám, Oral communication.
2. R. L. Davis, The Number of Structures of Finite Relations, *Proc. Amer. Math. Soc.* **4** (1953), 486–495.
3. J. Dénes, The Representation of a Permutation as the Product of a Minimal Number of Transpositions and Its Connection with the Theory of Graphs, *Publ. Math. Inst. Hungar. Acad. Sci.* **4** (1959), 63–71.
4. J. Dénes, On Transformations, Transformation-Semigroups and Graphs, *Proc. Colloq. Graph Theory, Tihany, 1966*, 65–75.
5. J. Dénes, Connections between Transformation Semigroups and Graphs, *Actes des Journées Internationals d'Étude sur la Théorie de Graphes, Rome, 1966*, Dunod, Paris, 1967, pp. 298–303.
6. P. Erdös and P. Turán, On Some Problems of Statistical Group Theory, I, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **4** (1965), 175–186. II, *Acta Math. Acad. Sci. Hung.* **18** (1967), 151–163. III, *Acta Math. Acad. Sci. Hung.* **18** (1967), 309–320, IV, *Acta Math. Acad. Sci. Hung.* **19** (1969), 413–435.
7. F. Harary, The Number of Functional Digraphs, *Math. Ann.* **138** (1959), 203–210.
8. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, Oxford, 1954.
9. L. Katz, Probability of Indecomposability of Random Mapping Function. *Ann. Math. Stat.* **26** (1953), 512–517.
10. O. Ore, *Theory of Graphs*, American Mathematical Society, Providence, R. I., 1962.
11. S. Piccard, *Sur les Bases du Groupe Symétrique*, I–II, Librairie Vuibert, Paris, 1946–48.
12. A. Rényi, Some Remarks on the Theory of Trees, *Publ. Math. Inst. Hungar. Acad. Sci.* **4** (1959), 73–85.
13. A. Rényi, On Connected Graphs, I, *Publ. Math. Inst. Hungar. Acad. Sci.* **4** (1959), 385–387.
14. S. M. Ulam, A Collection of Mathematical Problems, Wiley, New York, 1960.
15. D. Dixon, The probability of generating the symmetric graph. Preprint, Carleton University, Ottawa.
16. J. Dénes. Transformations and transformation semi-groups II. To appear in Hungarian.